

The University of Chicago

FIELDS FOR MULTIPLE
INTEGRALS IN THE CALCULUS OF
VARIATIONS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1934

By
BYRON COSBY II

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CONTRIBUTIONS TO THE CALCULUS OF VARIATIONS, 1933-37
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FIELDS FOR MULTIPLE INTEGRALS IN THE CALCULUS OF VARIATIONS

Introduction. It is the purpose of this paper to develop certain properties of fields and transversals of fields for the problem of minimizing a double integral of the form

$$I = \int_A \int f(x, y, z, p, q) dx dy$$

in a class of admissible surfaces $z = z(x, y)$, for each of which the integral I has a well-defined value, and all of which pass through a fixed boundary curve L in xyz -space. In this integral z denotes the unknown function $z(x, y)$, p and q stand for the derivatives z_x and z_y respectively, and A is the region of integration in the xy -plane.

The first mention of fields for the double integral problem seems to have been by Kneser [1, § 68].¹ His treatment was confined to the parametric case. Hilbert [2] discussed fields for the non-parametric problem in 1905. He gave a definition of a field in terms of a one-parameter family of extremals and gave certain conditions which the slope functions of a field must satisfy. Osgood [3] and Bolza [4] also discussed fields for the non-parametric problem and used the idea of fields in proving sufficiency theorems. The most comprehensive study of fields, as well as of the multiple integral problem, has been made by Bliss [5] in his lectures in the summer of 1933.

¹Numbers in brackets refer to the bibliography at the end of the paper.

The subject of fields in the simple integral problem is very closely related to the Hamilton-Jacobi theory. There have been several attempts to extend this theory to multiple integrals with only partial success. The first paper published on this subject was by Volterra [6] who made use of his then recently defined functional derivatives. His paper was confined to the study of the existence of extremals. Frechet [7] generalized this paper by extending the results to the case of an arbitrary number of independent variables.

Prange [8] has also written on the Hamilton-Jacobi theory. Like Volterra he was interested in the existence of extremals, but his paper also contains a discussion of certain functional fields. These fields, which are sets of extremals depending on an arbitrary function, do not seem to be of great value in any application of fields or in the study of transversals of fields. Born [9] has recently applied the methods of Prange to a problem in Quantum Mechanics, but nothing essentially new is added. De Donder [10] discussed the Hamilton-Jacobi theory with respect to canonical variables.

In this paper a study is made of the necessary and sufficient conditions for families of curves to be transversal curves of a field. These conditions depend upon the necessary and sufficient conditions for a pair of functions $p(x, y, z)$, $q(x, y, z)$ to be the slope functions of a field. The results here obtained parallel in some respects the Hamilton-Jacobi theory for simple integrals. It is necessary in the case of the double integral problem to have in addition to the Hamilton equation a second equation which enters because of the integrability condition on the functions $p(x, y, z)$ and $q(x, y, z)$ in the definition of a field. In the third section a form of the Hilbert integral

is deduced which depends only on two functions u and v which set equal to parameters define the transversal curves of the field. In the fourth section an analysis is made of an analogue of a theorem of Jacobi for the simple integral problem of the calculus of variations in the plane. For the problem in the plane this describes how the entire class of extremals may be determined, without further integration, when a complete integral of the Hamilton partial differential equation is known. In the double integral problem it seems that one cannot obtain the total class of extremals in such a manner. A discussion is also given in this section of the theorem of Jacobi as generalized by De Donder [11]. It can readily be shown, with the help of suitable changes in notation, that the results deduced in the following sections have generalizations for a problem of the calculus of variations in the space of points $(x_1, \dots, x_n, z)$ for which the integral to be minimized is an n -tuple instead of a double integral.

1. Fields and their Transversals. For analytic purposes the integrand $f(x, y, z, p, q)$ of the problem is assumed to be continuous and to possess continuous partial derivatives of the first and second orders in a region $\mathcal{R}$ of a five-dimensional space of points (x, y, z, p, q) . It is further assumed that the problem is regular, that is, that $f_{pp}f_{qq} - f_{pq}^2 > 0$ in $\mathcal{R}$. The class of admissible surfaces can be defined in a number of different ways. Throughout this paper a surface

$$z = z(x, y) \quad [(x, y) \text{ in } A]$$

will be said to be admissible if it is continuous, if it has continuous derivatives z_x and z_y , and if all of its elements (x, y, z, p, q) are interior to $\mathcal{R}$. The simply closed boundary

curve L is restricted to be continuous, to have continuous derivatives, and to be such that no two of its points lie on the same parallel to the z -axis.

The first definitions of fields were in terms of one-parameter families of extremals which simply covered a region $\mathcal{F}$. A more useful definition for the purposes of this paper was given by Bliss [5, p. 19]. An open bounded region $\mathcal{F}$ of the space of points (x, y, z) is said to be a field for the integral I if there is associated with it a pair of functions $p(x, y, z)$, $q(x, y, z)$ defined at each point of $\mathcal{F}$ and such that

- (1) they are continuous and possess continuous partial derivatives of the first and second orders, and the elements $[x, y, z, p(x, y, z), q(x, y, z)]$ for each point of $\mathcal{F}$ lie in $\mathcal{R}$;
- (2) they satisfy identically the equation

$$(1:1) \quad p_y + p_z \cdot q = q_x + q_z \cdot p ;$$

- (3) the Hilbert integral

$$(1:2) \quad I^* = \iint \left\{ f + (z_x - p)f_p + (z_y - q)f_q \right\} dx dy$$

has the same value on all surfaces in $\mathcal{F}$ with a common boundary curve. The arguments of f , f_p and f_q in this integral are $[x, y, z, p(x, y, z), q(x, y, z)]$.

Condition (3) of the definition implies [4, p. 659] that

$$(1:3) \quad \partial f_p / \partial x + \partial f_q / \partial y + \partial (p f_p + q f_q - f) / \partial z \equiv 0.$$

If the region $\mathcal{F}$ is simply connected, this last equation implies condition (3). By a simply connected region is meant one such that every closed surface interior to the region encloses only points of the region. Only simply connected regions are considered here.

Thus a necessary and sufficient condition that a pair of functions $p(x, y, z)$, $q(x, y, z)$ in a simply connected region be the slope functions of a field is that they satisfy the equations (1:1) and (1:3). These two equations are first order partial differential equations in the unknowns p and q . They are linear in the partial derivatives of p and q .

The identity (1:1) is a necessary and sufficient condition that the system of partial differential equations

$$(1:4) \quad \partial z / \partial x = p(x, y, z), \quad \partial z / \partial y = q(x, y, z)$$

be completely integrable [13, p. 424]. By completely integrable it is meant that in a neighborhood of each point (x_0, y_0, z_0) of the field $\mathcal{F}$ there exists a one-parameter family of solutions $z = z(x, y, \alpha)$ of (1:4) such that through each point of this neighborhood there passes one and but one member of the family. The function $z(x, y, \alpha)$ has continuous partial derivatives of the second order at least in a neighborhood of the values (x_0, y_0, α_0) defining the value z_0 , and has the further property that $z_{\alpha}(x, y, \alpha) \neq 0$ in this neighborhood.

An extremal is defined as an admissible surface with continuous second derivatives which satisfies the Euler-Lagrange equation

$$(1:5) \quad \partial f_p / \partial x + \partial f_q / \partial y - f_z = 0.$$

If the functions $p(x, y, z)$, $q(x, y, z)$ are the slope functions of a field, then the solutions of equations (1:4) are extremals. This statement can be verified by the differentiation of (1:3) [5, p. 20]. The resulting equation is

$$\begin{aligned}
 (\partial/\partial x)f_p + (\partial/\partial y)f_q + (\partial/\partial z)(pf_p + qf_q - f) = \\
 f_{px} + f_{pz}p + f_{pp}(p_x + p_zp) + f_{pq}(q_x + q_zp) + f_{qy} + f_{qz}q \\
 + f_{qp}(p_y + p_zq) + f_{qq}(q_y + q_zq) - f_z \equiv 0,
 \end{aligned}$$

which shows that the solutions of equations (1:4) are also solutions of the equation (1:5) of the extremals. Conversely it can be shown, at least for a field $\mathcal{F}$ bounded by a cylindrical surface whose generators are parallel to the z -axis, that every one-parameter family of extremals $z = z(x, y, \alpha)$ with the continuity properties in $\mathcal{F}$ previously described, and such that $z_\alpha(x, y, \alpha) \neq 0$ in $\mathcal{F}$, has a pair of slope functions $p(x, y, z)$, $q(x, y, z)$ which forms a field in $\mathcal{F}$ as defined above. Consequently the definition of a field in terms of properties of the slope functions $p(x, y, z)$, $q(x, y, z)$ as above given, is equivalent to the earlier definitions in terms of a one-parameter family of extremals.

A surface $\psi(x, y, z) = 0$ is said to be a transversal surface of a given field $\mathcal{F}$ if it satisfies the partial differential equation

$$(1:6) \quad f_p\psi_x + f_q\psi_y + (pf_p + qf_q - f)\psi_z = 0,$$

where the arguments of f , f_p and f_q are $[x, y, z, p(x, y, z), q(x, y, z)]$. It is easy to see that on every portion of one of these surfaces the invariant integral I^* has the value zero.

The transversal curves of a given field are by definition the characteristic curves of the differential equations (1:6). They satisfy the ordinary differential equations

$$(1:7) \quad dx/f_p = dy/f_q = dz/(pf_p + qf_q - f).$$

One and only one of these curves passes through each point of the

field. Every transversal surface is the locus of the transversal curves which intersect a given fixed curve in xyz-space, or is made up of portions each of which has this property.

The Hilbert invariant integral (1:2) can be reduced to a line integral by means of Stokes's Theorem. The integral has the form

$$I^* = \iint \left\{ f_p \, dy \, dz + f_q \, dx \, dz + (pf_p + qf_q - f) dx \, dy \right\}.$$

Since by the equation (1:3) the divergence of the vector

$[f_p, f_q, pf_p + qf_q - f]$ is zero, there exists another vector

$[A, B, C]$ whose components are functions of x, y, z , [12, p. 234], such that

$$\begin{aligned} (1:8) \quad C_y - B_z &= f_p, \\ A_z - C_x &= f_q, \\ B_x - A_y &= pf_p + qf_q - f. \end{aligned}$$

If $[A, B, C]$ is one solution of these equations, then the most general solution is $[A + \phi_x, B + \phi_y, C + \phi_z]$ where $\phi(x, y, z)$ is an arbitrarily selected function of x, y, z . From the last equations and by Stokes's Theorem

$$\begin{aligned} (1:9) \quad I^* &= \int_S \int \left\{ (C_y - B_z) dy \, dz + (A_z - C_x) dx \, dz + (B_x - A_y) dx \, dy \right\} \\ &= \int_C A \, dx + B \, dy + C \, dz, \end{aligned}$$

where C is the curve bounding the surface S . In Section 3 this integral will be expressed in a form involving only a pair of functions $u(x, y, z)$, $v(x, y, z)$ which set equal to parameters define the transversal curves.

The original variables of the problem are (x, y, z, p, q) . The canonical variables (x, y, z, π, κ) are introduced by setting

$$(1:10) \quad \pi = f_p(x, y, z, p, q), \quad \kappa = f_q(x, y, z, p, q).$$

By implicit function theorems there exists a solution of these equations

$$(1:11) \quad p = P(x, y, z, \pi, \kappa), \quad q = Q(x, y, z, \pi, \kappa)$$

with the same continuity properties as f_p and f_q , and such that

$$(1:12) \quad \pi \equiv f_p(x, y, z, P, Q), \quad \kappa \equiv f_q(x, y, z, P, Q).$$

The Hamilton function $H(x, y, z, \pi, \kappa)$ is defined as the function

$$pf_p + qf_q - f$$

with p and q replaced by P and Q respectively. The Hamilton function thus has the form

$$(1:13) \quad H(x, y, z, \pi, \kappa) = \pi P + \kappa Q - f(x, y, z, P, Q).$$

By differentiation of this identity one finds without difficulty

$$(1:14) \quad \begin{aligned} H_\pi &= P, & H_\kappa &= Q, & H_z &= -f_z, \\ H_{\pi\pi} &= P_\pi = f_{qq}/D, & H_{\pi\kappa} &= P_\kappa = Q_\pi = -f_{pq}/D, \\ H_{\kappa\kappa} &= Q_\kappa = f_{pp}/D, \end{aligned}$$

where $D = f_{pp}f_{qq} - f_{pq}^2$. In the customary way, by use of the Hamilton function, a canonical system of equations equivalent to the Euler-Lagrange equation can be defined. The canonical system has the form

$$(1:15) \quad \begin{aligned} \partial z / \partial x &= H_\pi(x, y, z, \pi, \kappa), & \partial z / \partial y &= H_\kappa(x, y, z, \pi, \kappa), \\ \partial \pi / \partial x &+ \partial \kappa / \partial y &= -H_z(x, y, z, \pi, \kappa). \end{aligned}$$

It is easy to prove the following theorem:

For every solution $z(x, y)$, $\pi(x, y)$, $\kappa(x, y)$ of the equations (1:15), with functions z, π, κ which are continuous and have continuous first and second partial derivatives, the function $z(x, y)$ defines an extremal surface satisfying the Euler-Lagrange equation (1:5). Conversely every extremal surface $z(x, y)$ defines by means of equations (1:10) a solution $z(x, y)$, $\pi(x, y)$, $\kappa(x, y)$ of the equations (1:15).

2. Systems of Curves Transversal to a Field. The transversal curves of a field are by definition the characteristic curves of the differential equation (1:6). They may be determined by equations of the form

$$(2:1) \quad u(x, y, z) = a, \quad v(x, y, z) = b$$

where u and v are solutions of equation (1:6) and have a matrix of first order determinants which are everywhere of rank two. We are interested here in the determination of the conditions under which the curves of a two-parameter family (2:1) are the transversal curves of a field. Differentiation of equations (2:1) gives for the direction $dx:dy:dz$ of the curve of intersection of the surfaces (2:1) the equations

$$u_x dx + u_y dy + u_z dz = 0,$$

$$v_x dx + v_y dy + v_z dz = 0.$$

In order to shorten many expressions which occur later, let us use the notations

$$(2:2) \quad \pi \equiv u_y v_z - u_z v_y, \quad \kappa \equiv u_z v_x - u_x v_z, \quad \rho \equiv u_x v_y - u_y v_x.$$

According to the theory of linear equations, and with this abridged notation, we have for the direction of the transversal

curves

$$dx = k\pi, \quad dy = k\kappa, \quad dz = k\rho$$

where $k(x, y, z)$ is a factor of proportionality, and π, κ, ρ do not vanish simultaneously. If one refers to equations (1:7), it is apparent that the following theorem is true.

THEOREM 2:1. A necessary and sufficient condition that the curves (2:1) be a set of curves transversal to a field whose slope functions $p(x, y, z)$, $q(x, y, z)$ are known, is that there exists a function $k(x, y, z)$ such that

$$(2:3) \quad \pi = kf_p, \quad \kappa = kf_q, \quad \rho = k(pf_p + qf_q - f),$$

where the arguments of f and its derivatives are $[x, y, z, p(x, y, z), q(x, y, z)]$.

This condition can be expressed in a form independent of $k(x, y, z)$, $p(x, y, z)$, $q(x, y, z)$. For the purpose of eliminating $k(x, y, z)$ the auxiliary theorem stated below is useful.

Auxiliary Theorem: If $u(x, y, z)$, $v(x, y, z)$ are two functions which satisfy equations (2:3) with the slope functions of a field and have a matrix of first derivatives everywhere of rank two, then there is always a pair of functions $U = G_1(u, v)$, $V = G_2(u, v)$ for which the equations

$$(2:4) \quad \begin{aligned} U_y V_z - U_z V_y &= f_p, & U_z V_x - U_x V_z &= f_q, \\ U_x V_y - U_y V_x &= pf_p + qf_q - f, \end{aligned}$$

hold with the same slope functions.

If u and v satisfy (2:3), they are solutions of (1:6), since by the definition of π, κ, ρ the equations

$$\pi u_x + \kappa u_y + \rho u_z = 0 ,$$

$$\pi v_x + \kappa v_y + \rho v_z = 0$$

are identities. Also from the definition of π, κ, ρ it follows without difficulty that

$$\pi_x + \kappa_y + \rho_z \equiv 0.$$

Let the three equations of the system of equations (2:3) be differentiated with respect to x, y, z , respectively. If the resulting equations are added one has the following identity:

$$\begin{aligned} \pi_x + \kappa_y + \rho_z &= k[(\partial/\partial x)f_p + (\partial/\partial y)f_q + (\partial/\partial z)(pf_p + qf_q - f)] \\ &\quad + k_x f_p + k_y f_q + k_z(pf_p + qf_q - f). \end{aligned}$$

Since the left side and the first member of the right side of this identity are zero, the remaining members of the right side total zero. Consequently $k(x, y, z)$ is a solution of (1:6). Since every solution of equation (1:6) is a function of the two independent solutions u, v , it follows that k is expressible in the form $k = K(u, v)$. Select $G_1(u, v), G_2(u, v)$ so that $\partial(G_1 G_2)/\partial(u, v) = 1/K$. Then

$$\begin{vmatrix} U_y & U_z \\ V_y & V_z \end{vmatrix} = \frac{\partial(G_1, G_2)}{\partial(u, v)} \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} = \frac{1}{K} \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix}$$

with the other similar determinants imply that equations (2:4) are satisfied.

Solutions $u(x, y, z), v(x, y, z)$ of equations (2:3) for different functions $k(x, y, z)$ represent the same family of curves. The above theorem states that it is possible to represent every such family in a form which satisfies (2:3) for $k = 1$. In the following paragraphs we shall seek first of all such representations.

In the equations

$$\begin{aligned}
 (2:5) \quad \pi &= f_p(x, y, z, p, q), \\
 \kappa &= f_q(x, y, z, p, q), \\
 \rho &= pf_p + qf_q - f
 \end{aligned}$$

let the variables (x, y, z, p, q) be considered as independent. As was shown in Section 1 the first two of these equations have solutions (1:11) such that

$$\begin{aligned}
 (2:6) \quad p &= P(x, y, z, \pi, \kappa) = H_{\pi}, & q &= Q(x, y, z, \pi, \kappa) = H_{\kappa}, \\
 \pi &\equiv f_p(x, y, z, P, Q), & \kappa &\equiv f_q(x, y, z, P, Q).
 \end{aligned}$$

By means of the Hamilton function (1:13) the elimination of p and q from equations (2:5) gives

$$\rho = H(x, y, z, \pi, \kappa).$$

This is evidently a necessary condition that $u(x, y, z) = a$, $v(x, y, z) = b$ form a family of transversal curves of a field satisfying equations (2:3), but it is not sufficient. A set of necessary and sufficient conditions are given in the following theorem.

THEOREM 2:2. Every two-parameter family of curves

$$(2:1) \quad u(x, y, z) = a, \quad v(x, y, z) = b,$$

with the determinants

$$\pi = u_y v_z - u_z v_y, \quad \kappa = u_z v_x - u_x v_z, \quad \rho = u_x v_y - u_y v_x$$

not vanishing simultaneously, is a family of transversal curves of a field if it satisfies the equations

$$\rho = H(x, y, z, \pi, \kappa),$$

(2:7)

$$(\partial/\partial y)H_{\pi} + H_{\kappa}(\partial/\partial z)H_{\pi} = (\partial/\partial x)H_{\kappa} + H_{\pi}(\partial/\partial z)H_{\kappa},$$

where the partial derivatives are taken with respect to the variables x, y, z occurring explicitly and in the functions π, κ . The slope functions of the field are $p = H_{\pi}$, $q = H_{\kappa}$. Conversely, every two-parameter family of transversals of a field has a representation (2:1) which satisfies these conditions.

The theorem is proved as follows. If $\pi(x, y, z)$, $\kappa(x, y, z)$, $\rho(x, y, z)$ satisfy the first equation of (2:7), then there exist two functions $p(x, y, z) = H_{\pi}(x, y, z, \pi, \kappa)$, $q(x, y, z) = H_{\kappa}(x, y, z, \pi, \kappa)$, defined by equations (2:6), such that π, κ, ρ, p, q satisfy equations (2:3). Thus if $p(x, y, z)$, $q(x, y, z)$ are the slope functions of a field, the theorem is proved. In order for them to be the slope functions of a field it is necessary and sufficient that they satisfy equations (1:1) and (1:3). But equation (1:1) is the same as the last equation of (2:7), and from the definition of π, κ, ρ in terms of the u and v the equation

$$\pi_x + \kappa_y + \rho_z = 0$$

is an identity. Since $\pi = f_p$, $\kappa = f_q$, $\rho = pf_p + qf_q - f$, equation (1:3) is always satisfied, and hence the integral I^* is independent of the path.

The converse of the theorem is true by virtue of the auxiliary theorem stated above.

Theorem 2:2 is an analogue of a theorem for the problem of the calculus of variations in the plane. The two functions $u(x, y, z)$, $v(x, y, z)$ here correspond to the function $W(x, y)$ [4, p. 128] which set equal to a parameter defines the transversal

curves for the plane problem. The determinants π , κ , ρ are analogues of the partial derivatives $\partial W/\partial x$, $\partial W/\partial y$. It is well known that these partial derivatives satisfy the Hamilton equation

$$\partial W/\partial x = H(x, y, \partial W/\partial y).$$

Conversely every solution $W(x, y)$ of this equation set equal to a parameter defines a family of transversal curves of the field characterized by the slope function $p(x, y) = H_\pi(x, y, \partial W/\partial y)$. The problem in the plane is simpler because no integrability conditions are needed in the definition of a field.

COROLLARY. If π , κ are two functions of x, y, z which satisfy the equations

$$(2:8) \quad \begin{aligned} \pi_x + \kappa_y + H_z + H_\pi \pi_z + H_\kappa \kappa_z &= 0, \\ (\partial/\partial y)H_\pi + H_\kappa(\partial/\partial z)H_\pi &= (\partial/\partial x)H_\kappa + H_\pi(\partial/\partial z)H_\kappa, \end{aligned}$$

and if ρ is defined in terms of them by the equation

$$\rho = H(x, y, z, \pi, \kappa),$$

then the vector field defined by $[\pi, \kappa, \rho]$ is the field of tangents to a two-parameter family of transversal curves of a field for the integral I. There exist two functions $u(x, y, z)$, $v(x, y, z)$ related to π, κ, ρ by the equations (2:2), and such that the transversal curves are defined by the equations $u = a$, $v = b$.

If π, κ satisfy the first of the equations (2:8), and if ρ is defined by the first of the equations (2:7), then it follows readily from the first equation (2:8) that the equations $\pi_x + \kappa_y + \rho_z = 0$ is an identity in x, y, z . Furthermore the slope functions defined by the equations $p = H_\pi$, $q = H_\kappa$ satisfy with π, κ, ρ the equations (2:5), and consequently the equation

(1:3). From the last equation of (2:8) these slope functions also satisfy the integrability condition (1:1). Hence they are the slope functions of a field for which the vectors $[\pi, \kappa, \rho]$ are the tangent vectors of the transversal curves. Every pair of independent solutions u, v of the equations

$$u_x \pi + u_y \kappa + u_z \rho = 0, \quad v_x \pi + v_y \kappa + v_z \rho = 0$$

has determinants of its derivatives proportional to π, κ, ρ . By the argument used in proving the auxiliary theorem of page 10 there exist functions $U = G_1(u, v)$, $V = G_2(u, v)$ whose determinants are exactly equal to π, κ, ρ , as indicated in equations (2:2).

The equations (2:8) are linear in the first partial derivatives of the functions π and κ . The determinant of coefficients of π_z, κ_z is readily found, with the help of equations (1:14), to be

$$-H_{\pi\pi}H_{\kappa\kappa}^2 + 2H_{\pi\kappa}H_{\pi\kappa}H_{\kappa\kappa} - H_{\kappa\kappa}H_{\pi\pi}^2 = -(f_{pp}p^2 + 2f_{pq}pq + f_{qq}q^2)/D.$$

If this quadratic form is different from zero the equations can be solved for π_z and κ_z . If the integrand f of the integral I is analytic, the Cauchy-Kowalewski existence theorems show that there exist analytic solutions $\pi(x, y, z)$, $\kappa(x, y, z)$ in a neighborhood of a point (x_0, y_0, z_0) such that for $z = z_0$ the solutions π and κ equal preassigned arbitrary analytic functions of x and y in the same neighborhood of (x_0, y_0, z_0) .

3. Further Properties of the Hilbert Integral. In Section 1 it was shown that the Hilbert integral may be expressed in the form of a line integral

$$(3:1) \quad I^* = \int_L A \, dx + B \, dy + C \, dz,$$

where A, B, C are functions of x, y, z satisfying equations (1:8). An inverse problem is to determine conditions on the functions A, B, C such that a line integral of the form (3:1) is a Hilbert invariant integral of a field for the integral I . This is equivalent to determining conditions for the existence of slope functions $p(x, y, z), q(x, y, z)$ of a field such that equations (1:8) will hold for the functions A, B, C belonging to a preassigned integral (3:1).

THEOREM 3:1. If A, B, C are functions of x, y, z such that they satisfy the partial differential equations (2:7) with π, κ, ρ replaced by

$$(3:2) \quad \pi = C_y - B_z, \quad \kappa = A_z - C_x, \quad \rho = B_x - A_y,$$

then the integral (3:1) is a line integral expressing the value of the Hilbert invariant integral for a field whose slope functions are the functions $p = H_\pi, q = H_\kappa$, with π and κ replaced by the values given in (3:2).

The proof is easily made. By Theorem (2:2) it follows that $p = H_\pi, q = H_\kappa$ are the slope functions of a field such that

$$\pi = f_p, \quad \kappa = f_q, \quad \rho = pf_p + qf_q - f,$$

where $\rho = H(x, y, z, \pi, \kappa)$. If π, κ, ρ are replaced by their values (3:2), then equations (1:8) will be satisfied, and it is evident that the integral (3:1) is related to the field as described in the theorem.

The properties of transversal curves of fields set forth in the last section lead to another interesting form of the Hilbert integral. Equations (2:3) and (1:8) imply that for a given field

$$C_y - B_z = u_y v_z - u_z v_y ,$$

$$A_z - C_x = u_z v_x - u_x v_z ,$$

$$B_x - A_y = u_x v_y - u_y v_x .$$

The solutions of this set of equations for A, B, C have the form

$$A = (uv_x - vu_x)/2 + \phi_x ,$$

$$B = (uv_y - vu_y)/2 + \phi_y ,$$

$$C = (uv_z - vu_z)/2 + \phi_z ,$$

where ϕ is an arbitrary function of (x, y, z) . If these values are substituted in the line integral (3:1) expressing the value of the Hilbert integral, it becomes

$$\begin{aligned} I^* &= \int_L A \, dx + B \, dy + C \, dz \\ &= (1/2) \int_L (uv_x - vu_x) dx + (uv_y - vu_y) dy + (uv_z - vu_z) dz \\ &= (1/2) \int_L u \, dv - v \, du. \end{aligned}$$

If the individual members of this last integral are integrated by parts, the integral can also be put into the form

$$I^* = \int_L u \, dv = - \int_L v \, du.$$

Here u and v are the functions which define the transversal curves of the given field. In the problem of the calculus of variations in the plane the corresponding form of the Hilbert integral is $I^* = \int dW = W$, where $W(x, y)$ is the function which defines the transversal curves of the given field.

It was remarked on page 6 that from the definition of a transversal surface of a field it could be seen that on every portion of such a surface the invariant integral I^* has the value zero. Conversely, every surface on every portion of which the

integral I^* vanishes is a transversal surface of the field for which the integral I^* is defined. This result can also be described by saying that every surface on which the line integral (3:1) is independent of the path is a transversal surface of the field to which this integral belongs. This statement in the latter form is now to be proved. Let the surface on which the integral is independent of the path be expressed in the parametric form

$$(3:3) \quad x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

where the functions x, y, z are continuous and possess continuous partial derivatives of the first and second orders. For a line integral of the form (3:1) to take the value zero for every closed curve on a surface it is necessary and sufficient that it be independent of the path of integration between every two points on the surface. The line integral (3:1) on the surface represented by equations (3:3) has the form

$$I^* = \int (Ax_u + By_u + Cz_u)du + (Ax_v + By_v + Cz_v)dv.$$

For such an integral to be independent of the path on a simply connected portion of the surface (3:3) it is necessary and sufficient that

$$\partial(Ax_u + By_u + Cz_u)/\partial v = \partial(Ax_v + By_v + Cz_v)/\partial u.$$

If these indicated differentiations are performed and the resulting expression rearranged, the last equation has the form

$$(C_y - B_z)(y_u z_v - y_v z_u) + (A_z - C_x)(x_v z_u - x_u z_v) + (B_x - A_y)(x_u y_v - x_v y_u) = 0.$$

By equations (1:8) it is readily seen that this equation is

equivalent to the equation (1:6) defining a transversal surface, since

$$\psi_x = k(y_u z_v - y_v z_u), \psi_y = k(x_v z_u - x_u z_v), \psi_z = k(x_u y_v - x_v y_u),$$

where k is a function of proportionality and ψ is the function of x, y, z which set equal to zero gives the non-parametric representation of the surface (3:3).

In the preceding pages two problems for a double integral I have been considered which are analogous to the problem of determining the function $W(x, y)$ defining the transversal curves of a field for a simple integral of the calculus of variations in the plane. The function $W(x, y)$ for the plane problem may be regarded as determining the one-parameter family of transversal curves $W(x, y) = c$, or it may be regarded as a function which has the property that the difference $W(x_2, y_2) - W(x_1, y_1)$ gives the value of the invariant integral of the field taken along a curve in the field from (x_1, y_1) to (x_2, y_2) . The characteristic property of $W(x, y)$ in either case is that it satisfies the Hamilton partial differential equation. If we regard $W(x, y)$ as determining the transversal curves then the analogue for the double integral of the problem of determining $W(x, y)$ in the plane is the determination of a two-parameter family $u(x, y, z) = a, v(x, y, z) = b$, which are the transversal curves of a field. The characteristic properties of u and v are that they satisfy the two partial differential equations (2:7), one of which is of the first and the other of the second order. If, on the other hand, we regard $W(x, y)$ as a function determining the value of the invariant integral then the analogous problem for the double integral is the determination of a line integral (3:1) in terms of which the invariant double integral of a field can be evaluated.

Characteristic properties of the coefficients A, B, C of the line integral are in this case the partial differential equations (2:7) in which π, κ, ρ are now the expressions (3:2). One of these equations (2:7) is again of the first and the other of the second order in the coefficients A, B, C . For either of these multiple integral problems the pair of equations (2:7), with the proper interpretation of the variables, is the analogue of the partial differential equation of Hamilton for the plane problem.

4. An Analysis of the Hamilton-Jacobi Theory. For the simple integral problem of the calculus of variations there is a theorem of Jacobi which states that from a complete integral of the Hamilton partial differential equation the totality of extremals can be determined by differentiation. This theorem apparently does not generalize to multiple integrals. However there is an equation, fundamental to the theory of the plane problem, which does have an analogue in the multiple integral case. In order to make the meaning of this equation clear a short discussion of the Jacobi theorem for the simple integral problem in the plane is given here.

The theorem of Jacobi states that if $W(x, y, \beta)$ is a complete integral of the Hamilton equation

$$\partial W / \partial x + H(x, y, \partial W / \partial y) = 0$$

such that $\partial^2 W / \partial y \partial \beta \neq 0$, then the two-parameter family of extremals of the problem is defined by the equation $\partial W / \partial \beta = \alpha$, where α is a parameter. A very simple proof of this theorem is given in [4, p. 138]. A second proof, involving procedure analogous to that which is to be used for the multiple integral problem, is as follows. The existence of the integral $W(x, y, \beta)$ implies the

existence of a one-parameter family of fields with slope functions $p(x, y, \beta) \equiv H_{\pi}(x, y, W_y)$ such that

$$\partial W / \partial x = f(x, y, p) - p f_p(x, y, p), \quad \partial W / \partial y = f_p(x, y, p).$$

These equations in turn imply that

$$\partial^2 W / \partial x \partial \beta = - p f_{pp}(\partial p / \partial \beta), \quad \partial^2 W / \partial y \partial \beta = f_{pp}(\partial p / \partial \beta),$$

and also the further relation

$$\partial^2 W / \partial x \partial \beta = - p(\partial^2 W / \partial y \partial \beta).$$

Now the solutions $y = y(x, \alpha, \beta)$ of the equation $\partial W / \partial \beta = \alpha$ all satisfy the derivative of this equation with respect to x , namely, the equation

$$(4:1) \quad (\partial^2 W / \partial x \partial \beta) + y'(\partial^2 W / \partial y \partial \beta) = 0.$$

These solutions $y = y(x, \alpha, \beta)$ are extremals since from the last and next to last equations they must also satisfy the equation $y' = p$. They form the totality of extremals of the problem since they are a two-parameter family.

It is this equation (4:1) whose analogue for the double integral problem is sought here. Let $u(x, y, z, \alpha)$, $v(x, y, z, \alpha)$ be a family of solutions of the equations (2:7) of Theorem 2:2, depending upon n parameters $\alpha = (\alpha_1, \dots, \alpha_n)$, and such that the determinants $\partial(u, v) / \partial(\alpha_1, \alpha_k)$ do not vanish simultaneously in the region $\mathcal{F}$ for the range of values of the parameters defining the family. Such a family of solutions implies the existence of an n -parameter family of fields with slope functions $p(x, y, z, \alpha) \equiv H_{\pi}(x, y, z, \pi, \kappa)$ and $q(x, y, z, \alpha) \equiv H_{\kappa}(x, y, z, \pi, \kappa)$, where π and κ are the determinants (2:2). From the equations (2:5), satisfied by these determinants and the

slope functions p, q , we have for each α_i

$$(4:2) \quad \begin{aligned} \frac{\partial \pi}{\partial \alpha_i} &= \frac{\partial P}{\partial \alpha_i} f_{pp} + \frac{\partial q}{\partial \alpha_i} f_{pq}, & \frac{\partial K}{\partial \alpha_i} &= \frac{\partial P}{\partial \alpha_i} f_{qp} + \frac{\partial q}{\partial \alpha_i} f_{qq}, \\ \frac{\partial \rho}{\partial \alpha_i} &= p \left\{ \frac{\partial P}{\partial \alpha_i} f_{pp} + \frac{\partial q}{\partial \alpha_i} f_{pq} \right\} + q \left\{ \frac{\partial P}{\partial \alpha_i} f_{qp} + \frac{\partial q}{\partial \alpha_i} f_{qq} \right\}. \end{aligned}$$

It is apparent then that

$$(4:3) \quad \frac{\partial \rho}{\partial \alpha_i} = p \frac{\partial \pi}{\partial \alpha_i} + q \frac{\partial K}{\partial \alpha_i} \quad (i = 1, 2, \dots, n).$$

Now consider the system of partial differential equations

$$(4:4) \quad \frac{\partial \rho}{\partial \alpha_i} = \frac{\partial z}{\partial x} \frac{\partial \pi}{\partial \alpha_i} + \frac{\partial z}{\partial y} \frac{\partial K}{\partial \alpha_i} \quad (i = 1, 2, \dots, n).$$

It is this system of equations which is analogous to the equation (4:1). The determinant of the coefficients of p and q in a pair of equation (4:3) reduces with the help of equations (4:2) to

$$(f_{pp}f_{qq} - f_{pq}^2) \partial(u, v) / \partial(\alpha_1, \alpha_k),$$

and according to the hypothesis made above these determinants do not vanish simultaneously in the region $\mathcal{F}$. A comparison of equations (4:3) and (4:4) shows therefore that $z_x = p$, $z_y = q$. But p and q are the slope functions of an n -parameter family of fields and satisfy the integrability conditions (1:1). Consequently the equations $z_x = p$, $z_y = q$, or the equivalent equations (4:4), have an $(n + 1)$ -parameter family of solutions $z = z(x, y, \alpha, \beta)$ which define extremal surfaces. We have therefore proved the following theorem:

THEOREM 4:1. Let the equations

$$(4:5) \quad u(x, y, z, \alpha) = a, \quad v(x, y, z, \alpha) = b,$$

depending upon $n \geq 2$ parameters $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, define the transversal curves of an n -parameter family of fields over a

region $\mathcal{F}$ of xyz-space, and let them have determinants $\partial(u, v)/\partial(\alpha_1, \alpha_k)$ which do not vanish simultaneously in $\mathcal{F}$ for values of the parameters α_1 defining the family. Then the n equations

$$\frac{\partial \rho}{\partial \alpha_i} = \frac{\partial z}{\partial x} \frac{\partial \pi}{\partial \alpha_i} + \frac{\partial z}{\partial y} \frac{\partial \kappa}{\partial \alpha_i} \quad (i = 1, 2, \dots, n)$$

have an $(n + 1)$ -parameter family of solutions $z = z(x, y, \alpha, \beta)$ which define extremal surfaces. This family of solutions can be found by solving a suitably selected pair of the equations for $\partial z/\partial x$ and $\partial z/\partial y$ and integrating.

From the proof of the preceding theorem it follows without difficulty that when transversal curves of the form (4:5) are known the slope functions of the corresponding field can be found by solving a suitably selected pair of the equations (4:3). They are therefore completely determined by differentiations with respect to the parameters α and solving linear equations.

Attempts to generalize the Jacobi theorem have been made by several writers. From the standpoint of the theory of functionals such generalizations have been made by De Donder [10], Frechet [7], and Prange [8, p. 52]. No one of these or any other generalizations seem to be of any service in the difficult and in general unsolved problem of determining an extremal surface through a given boundary curve.

De Donder [11, p. 84] has generalized the theorem without the use of the theory of functionals. His theorem, subject to many hypotheses, determines solutions of the canonical equations (1:15) from a solution of a generalization of the Hamilton equation without further integration. According to De Donder an integral $U(x, y, z, c_1, c_2), V(x, y, z, c_1, c_2)$ of the equation

$$(4:6) \quad (\partial U/\partial x) + (\partial V/\partial y) + H(x, y, z, \partial U/\partial z, \partial V/\partial z) = 0$$

is said to be complete if there exists a function $z(x, y, c_1, \dots, c_6)$ depending on six constants and satisfying the four equations

$$(4:7) \quad \partial U/\partial c_1 = c_3, \quad \partial U/\partial c_2 = c_4, \quad \partial V/\partial c_1 = c_5, \quad \partial V/\partial c_2 = c_6,$$

where c_3, c_4, c_5, c_6 are arbitrary constants. His theorem is then as follows:

THEOREM. Let U, V be a complete integral of equation (4:6) such that

$$\begin{vmatrix} \partial^2 U/\partial c_1 \partial z & \partial^2 U/\partial c_2 \partial z \\ \partial^2 V/\partial c_1 \partial z & \partial^2 V/\partial c_2 \partial z \end{vmatrix} \neq 0.$$

Then a solution $z = z(x, y, c_1, \dots, c_6)$ of the equations (4:7), if such a solution exists, defines a pair of functions $\pi(x, y, c_1, \dots, c_6), \kappa(x, y, c_1, \dots, c_6)$ by means of the equations

$$(4:8) \quad \partial U/\partial z = \pi, \quad \partial V/\partial z = \kappa,$$

and the set of these functions z, π, κ so defined is a solution of the canonical equations (1:15).

To prove this let U and V be substituted in equation (4:6). The resulting identity gives by differentiation

$$(4:9) \quad \partial^2 U/\partial x \partial z + \partial^2 V/\partial y \partial z + H_z + H_\pi(\partial^2 U/\partial z^2) + H_\kappa(\partial^2 V/\partial z^2) = 0,$$

$$(4:10) \quad \partial^2 U/\partial x \partial c_1 + \partial^2 V/\partial y \partial c_1 + H_{c_1}(\partial^2 U/\partial z \partial c_1) + H_{\kappa}(\partial^2 V/\partial z \partial c_1) = 0,$$

where c_1 stands for c_1 or c_2 . Since the integral U, V is complete, we know that there is at least one function $z = z(x, y, c_1, \dots, c_6)$ satisfying (4:7) identically. Let this function be substituted

in (4:8). If the first equation of (4:8) is differentiated with respect to x , the second with respect to y , and the results added, one obtains

$$(4:11) \quad \frac{\partial^2 U}{\partial z \partial x} + \frac{\partial^2 V}{\partial z^2} \cdot \frac{\partial z}{\partial x} + \frac{\partial^2 V}{\partial z \partial y} + \frac{\partial^2 V}{\partial z^2} \cdot \frac{\partial z}{\partial y} = \frac{\partial \pi}{\partial x} + \frac{\partial \kappa}{\partial y}.$$

If $\partial U / \partial c_1$ is differentiated with respect to x , and $\partial V / \partial c_1$ with respect to y , and the results added, one obtains

$$(4:12) \quad \frac{\partial^2 U}{\partial c_1 \partial x} + \frac{\partial^2 U}{\partial c_1 \partial z} \cdot \frac{\partial z}{\partial x} + \frac{\partial^2 V}{\partial c_1 \partial y} + \frac{\partial^2 V}{\partial c_1 \partial z} \cdot \frac{\partial z}{\partial y} = 0.$$

If equations (4:12) are subtracted from equations (4:10) one finds the equation

$$(\partial^2 U / \partial c_1 \partial z)(H_{\pi} - \partial z / \partial x) + (\partial^2 V / \partial c_1 \partial z)(H_{\kappa} - \partial z / \partial y) = 0.$$

These equations for $i = 1, 2$, imply

$$\partial z / \partial x = H_{\pi}, \quad \partial z / \partial y = H_{\kappa}$$

since the determinant of the theorem is different from zero. If equation (4:9) is subtracted from equation (4:11), then by virtue of the last displayed equation we have

$$\partial \pi / \partial x + \partial \kappa / \partial y = -H_z.$$

These last three equations are the canonical equations (1:15).

There is a serious objection to the theorem. The definition of a complete integral is based upon the assumption that there exists at least one function $z(x, y, c_1, \dots, c_6)$ satisfying the four equations (4:7). The possibility of the introduction of arbitrary functions [11, pp. 86-87] to make the equations (4:7) compatible needs more explanation.

The point of the proof at which need for equations (4:7) enters is in establishing equations (4:12). If $\partial U / \partial z$, $\partial V / \partial z$

are replaced in (4:12) by π and K as defined in (4:8), which is in accord with the notations previously used in this paper, it will be seen that the equations (4:12) are identical with the equations (4:4) when the set $(\alpha_1, \dots, \alpha_n)$ in (4:4) is replaced by (c_1, c_2) . The second and fourth terms in equations (4:12) are, in fact, the right side of equation (4:4). From equations (4:6) and (4:8) and the definition of $\rho = H$ it follows that the sum of the first and third terms of equation (4:12) is $-\partial\rho/\partial c_1$. In order for the functions $z(x, y, c_1, \dots, c_6)$ of (4:7) to define extremals it is necessary that it shall satisfy equations (4:12) and hence also equations (4:4). Consequently the class of solutions obtained by this theorem of De Donder is a subclass of the solutions defined by the method of Theorem 4:1.

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